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Asymptotics in the cocharacter sequence of $A \otimes B$

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Dedicated to Amitai Regev on his 65th birthday

Abstract

This paper obtains upper and lower bounds for the asymptotic behavior of the codimension sequence and cocharacter sequence of $A \otimes B$ in terms of the asymptotic behavior of the codimensions and cocharacters of A and B .

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In Amitai's PhD thesis he proved the conjecture that the tensor product of two PI algebras is again a PI algebra. Just how the identities of A and B are related to those of $A \otimes B$ has been the subject of a number of papers since then. We are adding the current paper to that number in this volume in honor of Amitai's birthday.

0. Introduction

We study cocharacters of tensor products of PI algebras in characteristic zero. Our main focus will be on two invariants of the cocharacter sequence. The first is the exponential rate of growth. In [6] Giambruno and Zaicev proved that if A is any PI algebra in characteristic zero and if $c_n(A)$ represents the n th codimension of A then the limit $\lim_{n \rightarrow \infty} \sqrt[n]{c_n(A)}$ exists and is an integer denoted $e(A)$. If for two PI algebras A and B we know $e(A)$ and $e(B)$, what does that tell us about $e(A \otimes B)$? Regev proved that $c_n(A \otimes B) \leq c_n(A)c_n(B)$, and so $e(A \otimes B) \leq e(A)e(B)$. We can say more:

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Theorem 1. *The exponential growth $e(A \otimes B)$ satisfies:*

(i) *For any PI algebras A and B ,*

$$e(A) + e(B) - 1 \leq e(A \otimes B) \leq e(A)e(B)$$

and both the upper and lower bound can occur.

- (ii) *For a given algebra A , $e(A \otimes B) = e(A)e(B)$ for all B if and only if there exists a verbally prime algebra A_1 which satisfies all of the identities of A and $e(A) = e(A_1)$.*
- (iii) *Given an algebra A and a positive integer b , the minimum value of $\{e(A \otimes B) | e(B) = b\}$ occurs when B is the $b \times b$ upper triangular matrices over F .*

The main tool in the proof of (i), is that there exist prime product algebras A' and B' which satisfy all of the identities of A and B , respectively, and such that $e(A) = e(A')$ and $e(B) = e(B')$. This fact was proven in [5]. Since A' and B' satisfy the identities of A and B , it follows that $c_n(A' \otimes B') \leq c_n(A \otimes B)$, and so it suffices to prove the lower bound in the case of prime product algebras. This proof will be the main theorem in Section 1, in which we will also supply the necessary background from the work of Kemer, Giambruno and Zaicev, and Berele and Regev. This section gives a complete description of the tensor product of prime product algebras. It can be viewed as a generalization of [2].

The other subject of this paper will be the invariants ω_0 and ω_1 called the arm and leg width of the cocharacter and defined in [3]. These describe which partitions correspond to irreducible S_n -characters occurring with non-zero multiplicity in the cocharacter $\chi_n(A)$. Let

$$\chi_n(A) = \sum_{\lambda \in \text{Par}(n)} m_\lambda \chi^\lambda$$

and let $\text{Supp}(A)$ be the set of λ with $m_\lambda \neq 0$. Then, $\omega_0(A)$ is the smallest d such that there exists $\lambda \in \text{Supp}(A)$ with d rows of unrestricted length. In the language of hooks, it is the smallest d so that $\text{Supp}(A)$ is contained in the hook $H(d, k)$, for some k . Similarly, $\omega_1(A)$ is the smallest h such that there exist $\lambda \in \text{Supp}(A)$ with h columns of unrestricted length; or the smallest h such that $\text{Supp}(A)$ is contained in the hook $H(k, h)$, for some k . We call $\omega_0(A)$ and $\omega_1(A)$ the eventual arm width and eventual leg width of A .

Our treatment of $\omega_0(A \otimes B)$ and $\omega_1(A \otimes B)$ will be parallel to that of $e(A \otimes B)$. First, Regev proved in [9] that $\chi_n(A \otimes B) \leq \chi_n(A) \otimes \chi_n(B)$. This implies

$$\omega_0(A \otimes B) \leq \omega_0(A)\omega_0(B) + \omega_1(A)\omega_1(B)$$

and

$$\omega_1(A \otimes B) \leq \omega_0(A)\omega_1(B) + \omega_0(A)\omega_1(B).$$

Then, just as in Theorem 1, we also prove a sharp lower bound, and a criterion for when the upper bound is necessarily obtained.

Theorem 2. *The eventual arm and leg widths $\omega_i(A \otimes B)$, $i = 0, 1$ satisfy*

(i) *For any PI algebras A and B ,*

$$\begin{aligned} & \max\{\omega_0(A) + \omega_0(B) - 1, 2(\omega_1(A) + \omega_1(B) - 1)\} \\ & \leq \omega_0(A \otimes B) \leq \omega_0(A)\omega_0(B) + \omega_1(A)\omega_1(B) \end{aligned}$$

and

$$\begin{aligned} & \max\{\omega_0(A) + \omega_1(B) - 1, \omega_1(A) + \omega_0(B) - 1, 2(\omega_1(A) + \omega_1(B) - 1)\} \\ & \leq \omega_1(A \otimes B) \leq \omega_0(A)\omega_1(B) + \omega_0(A)\omega_1(B) \end{aligned}$$

and both the upper and lower bound occur.

- (ii) *For a given algebra A and for either $i = 0$ or 1 , $\omega_i(A \otimes B)$ is equal to the upper bound for all algebras B if and only if there exists a verbally prime algebra A_1 which satisfies all of the identities of A and $(\omega_0(A), \omega_1(A)) = (\omega_0(A_1), \omega_1(A_1))$.*
- (iii) *Given an algebra A and a positive integers b_0 and b_1 , we consider the minimum value of both $\omega_0(A \otimes B)$ and $\omega_1(A \otimes B)$ in three cases: when B is constrained by $\omega_0(B) = b_0$; when B is constrained by $\omega_1(B) = b_1$; and when B is constrained by $(\omega_0(B), \omega_1(B)) = (b_0, b_1)$. In the first case, both minimums occur when B is the $b_0 \times b_0$ upper triangular matrices over F ; in the second case, both minimums occur when B is the $b_1 \times b_1$ upper triangular matrices over the Grassmann algebra E ; and in the last case, both minimums occur when B is the direct sum of these two algebras.*

The proof of the lower bounds in this case is similar to the proof in the case of $e(A \otimes B)$. It follows from [3] that there exists algebras A' and B' which are direct sums of prime product algebras with the properties that $\omega_i(A) = \omega_i(A')$ and $\omega_i(B) = \omega_i(B')$, for $i = 0, 1$, and A' and B' satisfy the identities of A and B , respectively. So, it will suffice to prove the lower bounds in the special case that A' and B' are direct sums of prime product algebras. This proof will be the subject of Section 2.

The machinery of Sections 1 and 2 also allows us to construct a counterexample. For any PI algebra A , let $\text{pp}(A)$ denote the direct sum of all of the prime product algebras which satisfy the identities of A . Then the cocharacter sequence of $\text{pp}(A)$ is known to give good asymptotic information about the cocharacter sequence of A , in the sense that $e(\text{pp}(A)) = e(A)$ and $\omega_i(\text{pp}(A)) = \omega_i(A)$, for $i = 0, 1$. So, it is reasonable to hope that the cocharacter sequence of $\text{pp}(A) \otimes \text{pp}(B)$ will be close to that of $A \otimes B$. Our counterexample shows that this need not be the case. We construct an algebra A for which all three invariants of $\text{pp}(A) \otimes \text{pp}(A)$ are strictly smaller than those of $A \otimes A$.

In the third and last section, we turn from prime product algebras to arbitrary PI algebras and we prove the last two parts of both Theorems 1 and 2.

We conclude our introduction by acknowledging the assistance of S. Catiou who made many useful suggestions. As we will point out in the text, two key combinatorial lemmas are essentially his.

1. Exponential rates of growth

Definition 3. A PI algebra A is *verbally prime*, if whenever a product of polynomials in distinct variables $f(x_1, \dots, x_n)g(x_{n+1}, \dots, x_m)$ is an identity for A , then either f or g is an identity for A .

These algebras were first defined by Kemer. Kemer proved a number of important theorems about verbally prime algebras, some of which we now record. For an account of Kemer's work, see [8] or [1]. Kemer proved the following in [7]. See also [10].

Theorem 4 (Kemer). Every verbally PI algebra is PI equivalent to one of the following: $M_n(F)$, $M_{k,\ell}$ or $M_n(E)$. Moreover, the tensor product of two verbally prime algebras is again verbally prime, with the products given by:

$$\begin{aligned} M_n \otimes M_m(F) &\sim M_{nm}(F), & M_n(F) \otimes M_{k,\ell} &\sim M_{nk,n\ell}, \\ M_n(F) \otimes M_m(E) &\sim M_{nm}(E), & M_{k,\ell} \otimes M_{a,b} &\sim M_{ka+\ell b, kb+\ell a}, \\ M_{k,\ell} \otimes M_n(E) &\sim M_{n(k+\ell)}(E), & M_n(E) \otimes M_m(E) &\sim M_{nm,nm}. \end{aligned}$$

The exponential rates of growth of the verbally prime algebras are known, see [4].

Theorem 5. The exponential rates of growth of the verbally prime algebras are given by: $e(M_n(F)) = n^2$, $e(M_{k,\ell}) = (k + \ell)^2$ and $M_n(E) = 2n^2$. It follows from this and the previous theorem that if A and B are verbally prime, then $e(A \otimes B) = e(A)e(B)$.

One reason that verbally prime algebras are important is this theorem of Kemer:

Theorem 6 (Kemer). If A is any PI algebra, then there exist an algebra PI equivalent to A of the form

$$A_1 \oplus \cdots \oplus A_n + J,$$

where J is a nilpotent ideal and the A_i are verbally prime algebras with $A_i A_j = 0$ for $i \neq j$.

We will say that an algebra of this form has a Kemer decomposition, and we will assume henceforth that all of our algebras do. In this case, it is useful to consider which products $C_1 J C_2 J \cdots J C_m$ are non-zero, where each C_i equals a different A_{i_α} .

Definition 7. Let A have Kemer decomposition, as above. A *reduced* subalgebra of A is one of the form $C = C_1 \oplus \cdots \oplus C_m + J$, where each $C_i = A_{i_\alpha}$ and the i_α are all different, and $C_1 J C_2 J \cdots J C_m \neq 0$. If $n = m$ we say that A is a reduced algebra. If in addition, $C_i J C_j = 0$ for all $i > j$, we say that C is a prime product algebra. More generally, given verbally prime algebras C_1, \dots, C_m there is a unique prime product algebra (up to PI equivalence) with summands C_1, \dots, C_m and we denoted it $C_1 \circ \cdots \circ C_m$. As a special case, $C \circ \cdots \circ C$ will be PI equivalent to $n \times n$ upper triangular matrices over C , which we denote $UT_n(C)$.

Corollary 8 (Kemer). *Every PI algebra is PI equivalent to a direct sum of reduced algebras. In particular, if A has a Kemer decomposition, then it is PI equivalent to the direct sum of its maximal reduced subalgebras.*

Proof. Let A have Kemer decomposition $A_1 \oplus \cdots \oplus A_n + J$. If f is any identity for A then f is an identity for every subalgebra of A , and so for the direct sum of the maximal reduced subalgebras. Conversely, assume that $f = f(x_1, \dots, x_k)$ is multilinear and not an identity for A . Then f is non-zero under a substitution in which each x_i takes values a_i from either J or some A_α . For some permutation σ ,

$$a_{\sigma(1)} \cdots a_{\sigma(k)} \neq 0.$$

It follows that the a_i are in a reduced subalgebra and hence f is not an identity for some reduced subalgebra. \square

In general, the cocharacter of a direct sum satisfies the bounds

$$\max\{\chi_n(A), \chi_n(B)\} \leq \chi_n(A \oplus B) \leq \chi_n(A) + \chi_n(B)$$

and so

$$e(A \oplus B) = \max\{e(A), e(B)\}. \quad (1)$$

It follows that $e(A)$ is equal to the maximum value of $e(A')$, where A' runs over the maximal reduced subalgebras. The value of $e(A')$ was computed by Giambruno and Zaicev, see [6].

Theorem 9 (Giambruno and Zaicev). *Let $C = C_1 \oplus \cdots \oplus C_m + J$ be a reduced algebra. Then $e(C) = e(C_1) + \cdots + e(C_m)$.*

This key lemma is an easy observation:

Lemma 10. *Let A and B be PI algebras with Kemer decompositions*

$$A = A_1 \oplus \cdots \oplus A_n + J(A) \quad \text{and} \quad B = B_1 \oplus \cdots \oplus B_m + J(B).$$

Then $A \otimes B$ has Kemer decomposition $\bigoplus_{i,j} A_i \otimes B_j + J$, where $J = A \otimes J(B) + J(A) \otimes B$.

The last ingredient we will need is a theorem of Berele and Regev, see [5].

Theorem 11 (Berele and Regev). *If A is any PI algebra, then there exists a prime product algebra A' which satisfies the identities of A such that $e(A) = e(A')$.*

As indicated in the introduction, we may now restrict attention to the case in which A and B are prime product algebras. Assume that A and B have Kemer decompositions as in Lemma 10.

Definition 12. We order $\mathbb{N} \times \mathbb{N}$ via $(a, b) < (c, d)$ if $a \leq c$, $b \leq d$ and $(a, b) \neq (c, d)$. Given $(n, m) \in \mathbb{N} \times \mathbb{N}$ we let $\mathcal{P}(n, m)$ be the set of all maximal chains from $(1, 1)$ to (n, m) , i.e., the set of all sequences I of the form

$$(1, 1) = (i_1, j_1) < \cdots < (i_t, j_t) = (n, m),$$

where $t = n + m - 1$ and each $(i_s, j_s) - (i_{s-1}, j_{s-1})$ is either $(1, 0)$ or $(0, 1)$. Given such an I and A and B prime product algebras, as above, we let $I(A, B)$ be the subalgebra of $A \otimes B$

$$(A_{i_1} \otimes B_{j_1}) \oplus \cdots \oplus (A_{i_t} \otimes B_{j_t}) + J.$$

In the language of Definition 7, $I(A, B) = (A_{i_1} \otimes B_{j_1}) \circ \cdots \circ (A_{i_t} \otimes B_{j_t})$.

Lemma 13. *The maximal reduced subalgebras of $A \otimes B$ (where A and B are prime product algebras) are precisely the algebras $I(A, B)$. Moreover, if A and B are prime product algebras, then every reduced subalgebra of $A \otimes B$ is a prime product algebra.*

Proof. Given I , in order to show that $I(A, B)$ is reduced we will show that there exists $a_s \in A_{i_s}$, $b_s \in B_{j_s}$ and $x_s \in J$ such that

$$(a_1 \otimes b_1)x_1 \cdots x_{t-1}(a_t \otimes b_t) \neq 0.$$

By definition of prime product algebras, there exists $a'_i \in A_i$ and $y_i \in J(A)$; and $b_j \in B_j$ and $z_j \in J(B)$ such that

$$a_1 y_1 \cdots y_{n-1} a_n \neq 0, \quad \text{and} \quad b_1 z_1 \cdots z_{m-1} b_m \neq 0.$$

We use induction on $t = n + m - 1$ to prove that

$$a_1 y_1 \cdots y_{n-1} a_n \otimes b_1 z_1 \cdots z_{m-1} b_m$$

which is non-zero, belongs to $(A_1 \otimes B_1)J \cdots J(A_n \otimes B_m)$. By maximality, (i_1, j_1) is $(1, 1)$, and we may assume without loss of generality that $(i_2, j_2) = (2, 1)$. Then the above product can be factored as

$$(a_1 \otimes 1)(y_1 \otimes 1)(a_2 y_2 y_{n-1} a_n \otimes b_1 z_1 \cdots z_{m-1} b_m),$$

where 1 represents the unit in B_1 . Now we are done by induction.

Conversely, if $(A_{i_1} \otimes B_{j_1})J \cdots J(A_{i_s} \otimes B_{j_s}) \neq 0$ in $A \otimes B$, then $A_{i_1} A \cdots A A_{i_s} \neq 0$ and $B_{j_1} B \cdots B B_{j_s} \neq 0$. By definition of prime product algebra prime product algebra the sequences (i_α) and (j_β) are each non-decreasing, and since the $A_{i_\alpha} \otimes B_{j_\beta}$ are distinct, the (i_α, j_α) are an increasing sequence in $\mathbb{N} \times \mathbb{N}$. So a reduced subalgebra corresponds to a chain in $\mathbb{N} \times \mathbb{N}$ and a maximal reduced subalgebra will correspond to a maximal chain. \square

Corollary 14. *The tensor product $A \otimes B$ of two prime product algebras with Kemer decomposition as above is PI equivalent to the direct sum $\bigoplus_{I \in \mathcal{P}(n, m)} I(A, B)$.*

Corollary 15. *If A and B are prime product algebras, then every reduced subalgebra of $A \otimes B$ is a prime product algebra.*

Combining Corollary 14 with Giambruno and Zaicev's theorem gives an effective way to compute $e(A \otimes B)$. In order to focus on the completely combinatorial aspect of the problem, we generalize the situation.

Definition 16. Let $a = (a_1, \dots, a_n)$ and let $b = (b_1, \dots, b_m)$ be finite sequences of non-negative integers. For each $I \in \mathcal{P}(n, m)$ we define $e(a, b, I)$ to be $\sum \{a_i b_j \mid (i, j) \in I\}$, and then we define $e(a, b)$ to be the maximum value $\max_{I \in \mathcal{P}(n, m)} e(a, b, I)$.

Lemma 17. *Given prime product algebras A and B , let $a = (e(A_1), \dots, e(A_n))$ and $b = (e(B_1), \dots, e(B_m))$. Then $e(A \otimes B) = e(a, b)$.*

Proof. Combining Corollary 14, Eq. (1) and Giambruno and Zaicev's theorem gives

$$e(A \otimes B) = \max_{I \in \mathcal{P}(n, m)} (I(A, B)) = \max_{I \in \mathcal{P}(n, m)} e(a, b, I) = e(a, b). \quad \square$$

The next key combinatorial lemma is essentially due to S. Catiou.

Lemma 18. *Let $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_m)$ be sequences of non-negative integers. Assume that $a_i = a'_i + a''_i$, where $a'_i, a''_i \geq 1$, and let \tilde{a} be the sequence gotten from a by replacing a_i by a'_i, a''_i . Then $e(\tilde{a}, b) \leq e(a, b)$.*

Proof. Let $\tilde{I} \in \mathcal{P}(n+1, m)$. We will construct $I \in \mathcal{P}(n, m)$ and prove that $e(\tilde{a}, b, \tilde{I}) \leq e(a, b, I)$. This will show that

$$e(\tilde{a}, b) = \max_{\tilde{I} \in \mathcal{P}(n+1, m)} e(\tilde{a}, b, \tilde{I}) \leq \max_{J \in \mathcal{P}(n, m)} e(a, b, J) = e(a, b).$$

Write \tilde{I} as

$$\begin{aligned} (1, j_{1,1}) &< \dots < (1, j_{1,s_1}) < \\ (2, j_{2,1}) &< \dots < (2, j_{2,s_2}) < \\ &\vdots & \\ (n+1, j_{n+1,1}) &< \dots < (n+1, j_{n+1,s_{n+1}}). \end{aligned}$$

Note that by definition of $\mathcal{P}(n+1, m)$, $j_{i,s_i} = j_{i+1,1}$, i.e., the last second co-ordinate corresponding to i equals the first one corresponding to $i+1$. We construct I by deleting $(i+1, j_{i+1,1})$, the first pair with first co-ordinate $i+1$, and subtracting 1 from each subsequent first co-ordinate:

$$I = (1, j_{1,1}) < \dots < (i, j_{i,s_i}) < \widehat{(i+1, j_{i+1,1})} < (i, j_{i+1,2}) < \dots < (n, j_{n+1,s_{n+1}}).$$

We now compare $e(\tilde{a}, b, \tilde{I})$ with $e(a, b, I)$.

$$e(\tilde{a}, b, \tilde{I}) = \sum_{k < i} a_k \sum_{u=1}^{s_k} b_{k,u} + a'_i \sum_{u=1}^{s_i} b_{i,u} + a''_i \sum_{u=1}^{s_{i+1}} b_{i+1,u} + \sum_{k > i} a_k \sum_{u=1}^{s_{k+1}} b_{k+1,u},$$

$$e(a, b, I) = \sum_{k < i} a_k \sum_{u=1}^{s_k} b_{k,u} + a_i \left(\sum_{u=1}^{s_i} b_{i,u} + \sum_{u=2}^{s_{i+1}} b_{i+1,u} \right) + \sum_{k > i} a_k \sum_{u=1}^{s_{k+1}} b_{k+1,u}.$$

Since $a_i = a'_i + a''_i$,

$$e(a, b, I) - e(\tilde{a}, b, \tilde{I}) = a'_i \sum_{u=1}^{s_i-1} b_{i,u} + a'_i \sum_{u=2}^{s_{i+1}} b_{i+1,u} \geq 0$$

and this proves the lemma. \square

Corollary 19. *For fixed b , the minimal value of $e(a, b)$ subject to $\sum a_i = \alpha$ occurs when $a = (1, 1, \dots, 1)$ (α 1's). Hence, if we fix $\sum a_i = \alpha$ and $\sum b_j = \beta$, then the minimal value of $e(a, b)$ when each of a and b are all 1's, and this minimum is $a + b - 1$.*

Using this corollary we can now prove the needed lower bound in Theorem 1(i).

Theorem 20 (= Theorem 1(i)). *For any PI algebras A and B , the exponential growth $e(A \otimes B)$ satisfies:*

$$e(A) + e(B) - 1 \leq e(A \otimes B) \leq e(A)e(B)$$

and both the upper and lower bound occur.

Proof. Let A and B be any two PI algebras. If we are interested in minimizing $e(A \otimes B)$, then Theorem 11 implies that we may assume that both A and B are prime product algebras, $A = A_1 \circ \dots \circ A_n$, $B = B_1 \circ \dots \circ B_m$. So, $e(A \otimes B)$ will be determined by the sequences $(e(A_i))_i$ and $(e(B_j))_j$, as in Lemma 17. Next, by Corollary 19 we can minimize $e(A \otimes B)$ by making each $e(A_i)$ and each $e(B_j)$ equal to 1. Finally, this happens when each $A_i = B_j = F$. \square

2. Eventual arm and leg widths

For an algebra A we will use the notation $\omega(A)$ for the ordered pair $(\omega_0(A), \omega_1(A))$. We will use the usual dot product $(a, b) \cdot (c, d) = ac + bd$, and the involution $(a, b)^\circ = (b, a)$, so that

$$(a, b)^\circ \cdot (c, d) = (a, b) \cdot (c, d)^\circ = ad + bc.$$

Then Regev's upper bound on $\omega(A \otimes B)$ can be written as

$$\omega(A \otimes B) \leq (\omega(A) \cdot \omega(B), \omega(A) \cdot \omega(B)^\circ).$$

If A and B are both verbally prime then $\omega(A)$ and $\omega(B)$ are known, see [6]. It follows from Kemer's tensor product theorem, Theorem 4 that the upper bound is obtained for each $\omega(A \otimes B)$. We record this fact as a lemma.

Lemma 21. *For the three classes of verbally prime algebras the eventual arm and leg widths are given by: $\omega(M_n(F)) = (n^2, 0)$, $\omega(M_{k,\ell}) = (k^2 + \ell^2, 2k\ell)$ and $\omega(M_n(E)) = (n^2, n^2)$. Hence, by Theorem 4, if A and B are verbally prime, $\omega(A \otimes B) = (\omega(A) \cdot \omega(B), \omega(A) \cdot \omega(B)^\circ)$.*

The reader may note that for each verbally prime algebra $\omega_0(A) \geq \omega_1(A)$. In fact, this is true for all PI algebras, as we proved in [3]. The proof involves the computation of the eventual arm and leg widths of arbitrary reduced algebras.

Theorem 22 (Berele). *If $A = A_1 \circ \cdots \circ A_n + J$ is a reduced algebra, then $\omega(A) = \omega(A_1) + \cdots + \omega(A_n)$. This implies that for any PI algebra A , $\omega_0(A) \geq \omega_1(A)$.*

Combining this theorem with Corollary 14 gives an effective way to compute the eventual arm and leg widths of the tensor product of two prime product algebras.

Lemma 23. *Let $A = A_1 \circ \cdots \circ A_n$ and $B = B_1 \circ \cdots \circ B_m$ be prime product algebras. Then,*

$$\omega_0(A \otimes B) = \max_{I \in \mathcal{P}(n,m)} \sum_{(i,j) \in I} \omega(A_i) \cdot \omega(B_j)$$

and

$$\omega_1(A \otimes B) = \max_{I \in \mathcal{P}(n,m)} \omega(A_i) \cdot \omega(B_j)^\circ.$$

Before continuing with the proof of Theorem 2(i), we construct the example described in the introduction of an algebra A for which $A \otimes A$ and $\text{pp}(A) \otimes \text{pp}(A)$ have asymptotically different cocharacters.

Counterexample 24. *There is an algebra A with $e(A \otimes A) = 16$ and $\omega(A \otimes A) = (8, 8)$, yet $e(A' \otimes A') = 12$ and $\omega(A' \otimes A') = (6, 6)$, where A' is the sum of the prime product algebras satisfying the identities of A .*

Proof. For convenience of notation, we construct two isomorphic algebras A_1 and A_2 , instead of one algebra A . Each A_i will be $Ex_i \oplus Ey_i + J_i$, where x_i and y_i are orthogonal idempotents, and J_i is generated by j_i , and is nilpotent of degree 3. Then

$$A_1 \otimes A_2 = (E \otimes E)(x_1 \otimes x_2) \oplus (E \otimes E)(x_1 \otimes y_2) \oplus (E \otimes E)(y_1 \otimes x_2) \oplus (E \otimes E)(y_1 \otimes y_2) \\ + J.$$

Moreover, $A_1 \otimes A_2$ is a reduced algebra, because

$$(x_1 \otimes x_2)(x_1 \otimes j_2)(x_1 \otimes y_2)(j_1 \otimes y_2)(y_1 \otimes y_2)(y_1 \otimes j_2)(y_1 \otimes x_2) \neq 0.$$

Since $e(E \otimes E) = 4$ and $\omega(E \otimes E) = (2, 2)$, the claimed values of $e(A_1 \otimes A_2)$ and $\omega(A_1 \otimes A_2)$ follow.

On the other hand, since $A_i/J_i \cong E \oplus E$ and $EJ_iE \neq 0$, $A' = E \circ E$ is the unique maximum prime product algebra which satisfies the identities of A_i . Using Corollary 14 one can show that $(E \circ E) \otimes (E \circ E)$ is PI equivalent to $(E \otimes E) \circ (E \otimes E) \circ (E \otimes E)$. Hence, $e(A' \otimes A') = 12$ and $\omega(A' \otimes A') = (6, 6)$, as claimed. \square

Definition and Corollary 25. Let $D = \{(u, v) \mid u \geq v \geq 0, u \geq 1\}$ in \mathbb{N}^2 . Let $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_m)$ be sequences in D . For $I \in \mathcal{P}(n, m)$, let $\omega(a, b, I) = (\omega_0(a, b, I), \omega_1(a, b, I))$ equal $\sum_{(i,j) \in I} (a_i \cdot b_j, a_i \cdot b_j^\circ)$, and let each $\omega_i(a, b)$ be the maximum value of $\omega_i(a, b, I)$, for all such I , and let $\omega(a, b) = (\omega_0(a, b), \omega_1(a, b))$. Then the previous lemma implies that

$$\omega(A \otimes B) = \omega(a, b),$$

where $a = (\omega(A_1), \dots, \omega(A_n))$ and $b = (\omega(B_1), \dots, \omega(B_m))$.

The next lemma is analogous to Lemma 18 and is also essentially due to Catiou.

Lemma 26. Let $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_m)$ be sequences in D . Assume that $a_i = a'_i + a''_i$, where $a'_i, a''_i \in D$, and let \tilde{a} be the sequence gotten from a by replacing a_i by a'_i, a''_i . Then $e(\tilde{a}, b) \leq e(a, b)$.

Proof. The same as Lemma 18. \square

Let A and B be direct sums of prime product algebras, $A = A_1 \oplus \dots \oplus A_k$ and $B = B_1 \oplus \dots \oplus B_h$. Then for $\alpha = 0, 1$

$$\omega_\alpha(A \otimes B) = \max \omega_\alpha(A_i \otimes B_j).$$

Lemma 27. Let $a = (a_1, \dots, a_n)$ be a sequence in D . Then for fixed b_0 , the minimal value of both $\omega_0(a, b)$ and $\omega_1(a, b)$, subject to $\omega_0(b) \geq b_0$ occurs when $b = ((1, 0), \dots, (1, 0))$; and for fixed b_1 the minimal value of both $\omega_0(a, b)$ and $\omega_1(a, b)$, subject to $\omega_1(b) \geq b_1$ occurs when $b = ((1, 1), \dots, (1, 1))$.

Proof. By Lemma 26 the minimums will occur on sequences with all elements having first co-ordinate equal to 1. By definition of D , such elements will be either $(1, 0)$ or $(1, 1)$. Since $((1, 0), \dots, (1, 0))$ is the smallest such sequence with first co-ordinates summing to b_0 , and $((1, 1), \dots, (1, 1))$ is the smallest with second co-ordinates summing to b_1 , the lemma follows. \square

Given $d \in D$ we denote by d^a the sequence consisting of a d 's.

Lemma 28. $\omega((1, 0)^a, (1, 0)^b) = (a + b - 1, 0)$, $\omega((1, 0)^a, (1, 1)^b) = (a + b - 1, a + b - 1)$, $\omega((1, 1)^a, (1, 1)^b) = (2(a + b - 1), 2(a + b - 1))$.

Proof. Follows from a straightforward computation using the definition of ω . \square

Theorem 29 (= Theorem 2(i)). For fixed $d_1 \geq h_1, d_2 \geq h_2$, let A run over algebras with $\omega(A) = (d_1, h_1)$, and B run over algebras with $\omega(B) = (d_2, h_2)$. The minimum value of $\omega_0(A \otimes B)$ is $\max\{d_1 + d_2 - 1, 2(h_1 + h_2 - 1)\}$ and the minimum value of $\omega_1(A \otimes B)$ is $\max\{d_1 + h_2 - 1, d_2 + h_1 - 1, 2(h_1 + h_2 - 1)\}$.

Proof. It follows from [3] that there exists prime product algebras A_0 and A_1 satisfying the identities of A such that $\omega_i(A_i) = \omega_i(A)$, $i = 0, 1$; and likewise B_0 and B_1 . Hence $\omega(A_0 \oplus A_1) = \omega(A)$ and $\omega(B_0 \oplus B_1) = \omega(B)$, and

$$\omega((A_0 \oplus A_1) \otimes (B_0 \oplus B_1)) \leq \omega(A \otimes B).$$

Without loss of generality, we take $A = A_0 \oplus A_1$ and $B = B_0 \oplus B_1$.

Now, $A \otimes B = \bigoplus_{i,j} A_i \otimes B_j$. By Lemma 27 each eventual arm and leg width will be minimized by taking A_0 and B_0 equal to $F \circ \cdots \circ F$ (a_0 times for A_0 and b_0 times for B_0), and A_1 and B_1 equal to $E \circ \cdots \circ E$ (a_1 times for A_1 and b_1 times for B_1). The numerical value of the lower bound follows from the previous lemma. \square

3. Lower bounds

In this and the next section we prove various results on $A \otimes B$ without the assumption that A and B are prime product algebras. It will be useful to describe $A \otimes B$ when A is a reduced algebra and B is a prime product algebra. We generalize some of the machinery from Section 1.

Definition 30. Given n and m , let $\mathcal{Q}(n, m)$ be the set of all sequences of distinct pairs of positive integers $(i_1, j_1), \dots, (i_t, j_t)$ such that $1 \leq i_1 \leq \cdots \leq i_t \leq n$ and each j_a is between 1 and m . Note that the requirement that the elements of the sequence are distinct implies that if $i_a = i_b$, then $j_a \neq j_b$.

Lemma 31. Given a prime product algebra $A = A_1 \circ \cdots \circ A_n$ with radical $J(A)$ and a reduced algebra $B = B_1 \oplus \cdots \oplus B_m + J(B)$, let $I = (i_1, j_1), \dots, (i_t, j_t) \in \mathcal{Q}(n, m)$. Then

$$B_{j_1} B B_{j_2} B \cdots B B_{j_t} \neq 0 \Leftrightarrow (A_{i_1} \otimes B_{j_1}) J \cdots J (A_{i_t} \otimes B_{j_t}) \neq 0,$$

where $J = J(A) \otimes B + A \otimes J(B)$.

Proof. (\Leftarrow): Trivial.

(\Rightarrow): Let $b_{j_1} b'_{j_1} \cdots b'_{j_{t-1}} b_{j_t}$ be a non-zero element of $B_{j_1} B \cdots B B_{j_t}$, with obvious notation. For each α such that $i_\alpha \neq i_{\alpha+1}$, let $x_\alpha \in J(A)$ be such that $A_{i_\alpha} x_\alpha A_{i_{\alpha+1}} \neq 0$ and such that the product $x_{\alpha_1} \cdots x_{\alpha_s}$ is not zero. We claim that

$$0 \neq x_{\alpha_1} \cdots x_{\alpha_s} \otimes b_{j_1} b'_{j_1} \cdots b'_{j_{t-1}} b_{j_t}$$

is a non-zero element of $(A_{i_1} \otimes B_{j_1}) J \cdots J (A_{i_t} \otimes B_{j_t})$. The proof will be by induction on t , the case of $t = 1$ being trivial. We consider separately the cases in which $i_1 = i_2$ and $i_1 \neq i_2$.

If $i_1 = i_2$, then $j_1 \neq j_2$ and so the fact that $b_{j_1} b'_{j_1} b_{j_2}$ is not zero allows us to assume that b'_{j_1} is an element of $J(B)$. So,

$$x_{\alpha_1} \cdots x_{\alpha_s} \otimes b_{j_1} b'_{j_1} b_{j_2} \cdots b_{j_t} = (1 \otimes b_{j_1})(1 \otimes b'_{j_1})(x_{\alpha_1} \cdots x_{\alpha_s} \otimes b_{j_2} \cdots b_{j_t})$$

where the 1 denotes the identity element in A_{i_1} . Note that the first two factors are in $(A_{i_1} \otimes B_{j_1})J$, and that the third factor has a smaller value of t , and so we are done by induction.

Finally, if $i_1 \neq i_2$, then $\alpha_1 = 1$ and we can factor the element as

$$(1 \otimes b_{j_1})(x_1 \otimes b'_1)(x_{\alpha_2} \cdots x_{\alpha_s} \otimes b_{j_2} b'_2 \cdots b_{j_t})$$

and we are again done by induction. \square

Definition 32. Given A a prime product algebra and B a reduced algebra as above, let $\mathcal{Q}(n, B)$ be all of the elements of $\mathcal{Q}(n, m)$ such that $B_{j_1} B B_{j_2} B \cdots B B_{j_t} \neq 0$ as in the previous lemma. For $I \in \mathcal{Q}(n, B)$ let $I(A, B)$ be the subalgebra of $A \otimes B$, $(A_{i_1} \otimes B_{j_1}) \oplus \cdots \oplus (A_{i_t} \otimes B_{j_t}) + J$.

As an immediate consequence of Lemma 31 we get analogues of Lemma 13 and Corollary 14.

Lemma 33. The $I(A, B)$ are the reduced subalgebras of $A \otimes B$, and $A \otimes B$ is PI equivalent to the direct sum of the maximal $I(A, B)$.

Note that $e(I(A, B)) = \sum_{(i,j) \in I} e(A_i) e(B_j)$, and that $e(A \otimes B)$ will be the maximal such. This suggests the following definition.

Definition 34. Given $a = (a_1, \dots, a_n)$ a sequence of positive integers and $I \in \mathcal{Q}(n, B)$, let $e(a, B, I)$ equal $\sum_{(i,j) \in I} a_i e(B_j)$, and let $e(a, B)$ be the maximum value of $e(a, B, I)$. By the above, if $a = (e(A_{i_1}), \dots, e(A_{i_t}))$, then $e(a, B) = e(A \otimes B)$.

Here now is the analogue of Lemma 18.

Lemma 35. Let $a = (a_1, \dots, a_n)$ be a sequence of positive integers and B a reduced algebra. Assume that $a_i = a'_i + a''_i$, where $a'_i, a''_i \geq 1$, and let \tilde{a} be the sequence constructed from a by replacing a_i by a'_i, a''_i . Then $e(\tilde{a}, B) \leq e(a, B)$.

Proof. Let $\tilde{I} \in \mathcal{Q}(n+1, B)$. As in the proof of Lemma 18 we will construct $I \in \mathcal{Q}(n, B)$ such that $e(\tilde{a}, B, \tilde{I}) \leq e(a, B, I)$, and this will prove the lemma. To construct I we consider the part of \tilde{I} with first co-ordinate equal to i or $i+1$. Say they are $i = i_\alpha = \cdots = i_\beta$ and $i+1 = i_{\beta+1} = \cdots = i_\gamma$. By definition of $\mathcal{Q}(n, m)$, the second co-ordinates j_α, \dots, j_β are all distinct, and $j_{\beta+1}, \dots, j_\gamma$ are all distinct, but there may be overlap between these two sequences. Let k_1, \dots, k_a enumerate the overlap. We construct I from \tilde{I} by first eliminating the elements $(i+1, k_1), \dots, (i+1, k_a)$, and then decreasing every first co-ordinate greater than i by 1. We now compare $e(\tilde{a}, b, \tilde{I})$ with $e(a, b, I)$.

$$e(\tilde{a}, b, \tilde{I}) = \sum_{k < i} a_k \sum_{u=1}^{s_k} b_{k,u} + a'_i \sum_{u=1}^{s_i} b_{i,u} + a''_i \sum_{u=1}^{s_{i+1}} b_{i+1,u} + \sum_{k > i} a_k \sum_{u=1}^{s_{k+1}} b_{k+1,u},$$

$$e(a, b, I) = \sum_{k < i} a_k \sum_{u=1}^{s_k} b_{k,u} + a_i \left(\sum_{u=1}^{s_i} b_{i,u} + \sum_{u=1}^{s_{i+1}} b_{i+1,u} - \sum_{u=1}^a b_{i+1,k_u} \right) + \sum_{k > i} a_k \sum_{u=1}^{s_{k+1}} b_{k+1,u}.$$

Since $a_i = a'_i + a''_i$,

$$e(a, b, I) - e(\tilde{a}, b, \tilde{I}) = a''_i \sum_{u=1}^{s_i} b_{i,u} + a'_i \sum_{u=1}^{s_{i+1}} b_{i+1,u} - (a'_i + a''_i) \sum_{u=1}^a b_{i+1,k_u} \geq 0$$

and this proves the lemma. \square

Theorem 36 (= Theorem 1(iii)). Given a PI algebra B and an integer a , the minimal value of $e(A \otimes B)$, subject to $e(A) = a$, occurs when A is the $a \times a$ upper triangular matrices $U_a(F)$.

Proof. First recall that $U_a(F) = F \circ \dots \circ F$ and each $e(F) = 1$. Now, let A be any PI algebra with $e(A) = a$. By Theorem 11, there exists a prime product algebra A' with $e(A' \otimes B) \leq e(A \otimes B)$. Say $A' = A_1 \circ \dots \circ A_n$ and let a be the sequence $(e(A_1), \dots, e(A_n))$. Then, using Lemma 35

$$e(A' \otimes B) = e(a, B) \geq e((1, \dots, 1), B) = e(F \circ \dots \circ F, B). \quad \square$$

Theorem 36 and its proof generalize easily to $\omega(A \otimes B)$. Using Lemma 33 the subsequent definition and proofs may be adapted to eventual arm and leg widths.

Definition 37. Given $a = (a_1, \dots, a_n)$ a sequence of elements of D and $I \in \mathcal{Q}(n, B)$, let $\omega(a, B, I)$ equal $\sum_{(i,j) \in I} (a_i \cdot \omega(B_j), a_i \cdot \omega(B_j)^\circ)$. For $i = 0, 1$ let $\omega_i(a, B)$ be the maximum value of $\omega_i(a, B, I)$. It follows from Lemma 33 that if $a = (\omega(A_{i_1}), \dots, \omega(A_{i_r}))$, then $\omega(a, B) = \omega(A \otimes B)$.

The proof of the next lemma is the same as that of its analogue, Lemma 35.

Lemma 38. Let $a = (a_1, \dots, a_n)$ be a sequence of elements of D and B a reduced algebra. Assume that $a_i = a'_i + a''_i$, where $a'_i, a''_i \in D$, and let \tilde{a} be the sequence constructed from a by replacing a_i by a'_i, a''_i . Then $\omega_i(\tilde{a}, B) \leq \omega_i(a, B)$ for $i = 0, 1$.

Theorem 39. Given $(a_0, a_1) \in D$ and a PI algebra B

- (i) The minimal values of both $\omega_0(A \otimes B)$ and $\omega_1(A \otimes B)$ where A is constrained by $\omega_0(A) = a_0$ occur when A is the $a_0 \times a_0$ upper triangular matrices over F .
- (ii) The minimal values of both $\omega_0(A \otimes B)$ and $\omega_1(A \otimes B)$ where A is constrained by $\omega_1(A) = a_1$ occur when A is the $a_1 \times a_1$ upper triangular matrices over E .
- (iii) The minimal values of both $\omega_0(A \otimes B)$ and $\omega_1(A \otimes B)$ where A is constrained by $\omega_0(A) = a_0$ and $\omega_1(A) = a_1$ occur when A is the direct sum of the $a_0 \times a_0$ upper triangular matrices over F and the $a_1 \times a_1$ upper triangular matrices over E .

Proof. By [3] we may assume that A is a direct sum of prime product algebras, and by Lemma 35 we may assume that all of the factors in A are either F , which satisfies $\omega(F) = (1, 0)$ or E which satisfies $\omega(E) = (1, 1)$. But $F \circ \dots \circ F$, which is the upper triangular matrices, is the smallest prime product algebra with $\omega_0(A) = a_0$; and $E \circ \dots \circ E$, which is the upper triangular matrices over E , is the smallest prime product algebra with $\omega_1(A) = a_1$; and their direct sum is the smallest sum of prime product algebras with $\omega(A) = (a_0, a_1)$. \square

Theorem 40. For a given algebra A , $e(A \otimes B) = e(A)e(B)$ for all B if and only if there exists a verbally prime algebra A_1 which satisfies all of the identities of A and $e(A) = e(A_1)$.

Moreover, for a given algebra A , the following are equivalent:

- (i) There exists a verbally prime algebra A_1 which satisfies all of the identities of A and $\omega(A) = \omega(A_1)$.
- (ii) For every B , $\omega_0(A \otimes B) = \omega_0(A)\omega_0(B) + \omega_1(A)\omega_1(B)$.
- (iii) For every B , $\omega_1(A \otimes B) = \omega_0(A)\omega_1(B) + \omega_1(A)\omega_0(B)$.

Proof. First, assume there exists a verbally prime A_1 that satisfies all of the identities of A and has the same exponential rate of growth, and let B be any PI algebra. Then by Corollary 8 we may assume that B is a reduced algebra, $B = B_1 \oplus \cdots \oplus B_n + J$. So $A_1 \otimes B = (A_1 \otimes B_1) \oplus \cdots \oplus (A_1 \otimes B_n) + (A_1 \otimes J)$ and

$$\begin{aligned} e(A \otimes B) &\geq e(A_1 \otimes B) = e(A_1 \otimes B_1) + \cdots + e(A_1 \otimes B_n) \\ &= e(A_1)e(B_1) + \cdots + e(A_1)e(B_n) = e(A)(e(B_1) + \cdots + e(B_n)) \\ &= e(A)e(B). \end{aligned}$$

If we assume that there exists a verbally prime A_1 such that $\omega(A) = \omega(A_1)$, the computation of $\omega(A \otimes B)$ is similar. For example, in the case of $\omega_0(A \otimes B)$ we have

$$\begin{aligned} \omega_0(A \otimes B) &\geq \omega_0(A_1 \otimes B) = \omega_0(A_1 \otimes B_1) + \cdots + \omega_0(A_1 \otimes B_n) \\ &= \omega_0(A_1) \cdot \omega_0(B_1) + \cdots + \omega_0(A_1) \cdot \omega_0(B_n) \\ &= \omega_0(A) \cdot (\omega_0(B_1) + \cdots + \omega_0(B_n)) \\ &= \omega_0(A) \cdot \omega_0(B). \end{aligned}$$

To prove the converse, we will show that if A is not verbally prime, then $e(A \otimes B) < e(A)e(B)$ and $\omega(A \otimes B) < (\omega(A) \cdot \omega(B), \omega(A) \cdot \omega(B)^\circ)$, where $B = B_1 \circ \cdots \circ B_r$, where each B_j is isomorphic to E and where r is big enough. By Corollary 8 we may assume that A is a reduced algebra with Kemer decomposition $A = A_1 \oplus \cdots \oplus A_n + J(A)$ and that $J(A)^q = 0$ and $n \geq 2$. Consider the Kemer decomposition of $A \otimes B$ as in Lemma 10. A non-zero reduced subalgebra of $A \otimes B$ will be of the form

$$(A_{1,i_1} \otimes B_1)J \cdots J(A_{1,i_{s_1}} \otimes B_1)J(A_{2,i_2} \otimes B_2)J \cdots$$

We denoted each subproduct $(A_{k,i_1} \otimes B_k)J \cdots J(A_{k,i_{s_k}} \otimes B_k)$ with second factor equal to B_k as P_k . In each P_k the $A_{k,i}$ are all different. Now, since $J = J(A) \otimes B + A \otimes J(B)$, and since the different A_i are orthogonal, we may assume within each P_k that each J is of the form $J(A) \otimes B$. Since $J(A)^q = 0$ there will be a total of at most $q - 1$ of the P_k with two or more A_i in the product. Within these, the sum of the $e(A_i)$ is at most $e(A)$. By hypothesis, for the remaining $r - q + 1$ values of k , the $e(A_i)$ in the product is at most $e(A) - 1$. Keeping in mind that $e(E) = 2$, we have

$$\begin{aligned} e(A \otimes B) &\leq 2(q - 1)e(A) + 2(r - q + 1)(e(A) - 1) \\ &< 2re(A) = e(B)e(A). \end{aligned}$$

Finally, we turn to the eventual arm and leg widths. Given A_i verbally prime with $\omega(A_i) = (a, b)$, $\omega(A_i \otimes E) = (a + b, a + b)$. We write $a + b$ as $|(a, b)|$, and note that if $(a, b) < (c, d)$ then $|(a, b)| < |(c, d)|$. Then, as in the case of $e(A \otimes B)$,

$$\begin{aligned}\omega_i(A \otimes B) &\leq (q - 1)|\omega(A)| + (r - q + 1)(|\omega(A)| - 1) < r|\omega(A)| \\ &= \omega_0(B)\omega_0(A) + \omega_1(B)\omega_1(A) = \omega_1(B)\omega_0(A) + \omega_0(B)\omega_1(A). \quad \square\end{aligned}$$

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